

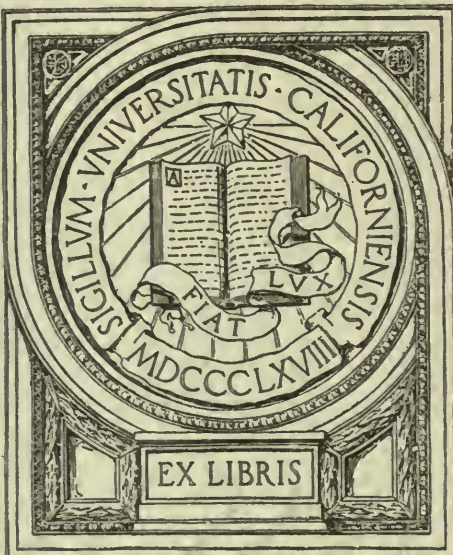
QA
601
L5

UC-NRLF



QC 168 190

EXCHANGE

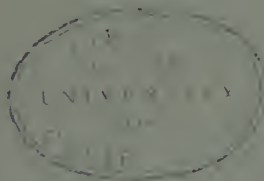


EX LIBRIS

CONTRIBUTIONS TO EQUILONG GEOMETRY

BY

PAUL HENRY LINEHAN



SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY
OF PURE SCIENCE, COLUMBIA UNIVERSITY

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

1915

CONTRIBUTIONS TO EQUILONG GEOMETRY

BY

PAUL HENRY LINEHAN

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY, IN THE FACULTY
OF PURE SCIENCE, COLUMBIA UNIVERSITY



PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

1915

QAG01
L5

EXCHANGE

70 1111
11111111

CONTENTS.

INTRODUCTION.....	Page v
-------------------	-----------

CHAPTER I.

The Invariants of Irregular Analytic Curves under the Group of Equilong Transformations of the Plane.....	1
--	---

CHAPTER II.

Some Curves with Equations in Hessian Line Coordinates.

Sect. 1. Hessian Line Coordinates.....	4
2. Evolutes and Involutives.....	5
3. Radii of Curvature.....	8
4. Pedals and Negative Pedals.....	9

CHAPTER III.

Linear Equilong Transformations.

Sect. 1. The Integral Linear Transformations.....	12
(a) The Operation $W = w + \beta$	12
(b) The Operation $W = \alpha w$	15
(c) The Operation $W = \alpha w + \beta$	19
Sect. 2. The General Linear Transformations.....	21
(a) The Operation $W = \frac{1}{w}$	21
(b) Combinations of the Operations $W = w + \beta$ and $W = \frac{1}{w}$	25
(c) Combinations of the Operations $W = \alpha w + \beta$ and $W = \frac{1}{w}$	27
(d) The Fixed Lines and the Criteria for the Types of Motion.....	31
(e) The Simplest Invariant of a Regular Analytic Curve.....	35

327802



Digitized by the Internet Archive
in 2008 with funding from
Microsoft Corporation

INTRODUCTION.

The infinite group of equilong transformations of the plane* is that group of contact transformations of the plane by which a straight line becomes a straight line and distance on the line is unchanged. Two arbitrary lineal elements of one and the same line become, under an equilong transformation, two lineal elements of the transformed line, the distance between the points of the original elements being equal to the distance between the points of the new elements.

If the lines of the plane be determined by the Hessian coordinates (u, v) , v being the length of a perpendicular from a fixed point to the line and u being the angle between the perpendicular and a fixed line through the fixed point,† and if the lines of the transformed plane be determined by the Hessian coordinates (U, V) , the infinite group of equilong transformations will be represented by

$$U + jV = f(u + jv),$$

$f(u + jv)$ denoting an arbitrary analytic function of the so-called dual number $u + jv$ of a system with the unities 1 and j in which $j^2 = 0$.

The equilong transformations may be expressed by

$$U = \varphi(u),$$

$$V = v \cdot \frac{d\varphi(u)}{du} + \chi(u),$$

in which φ and χ are arbitrary analytic functions of u .

In this article several topics in equilong geometry are treated.

Chapter I is concerned with the existence and derivation of the invariants of irregular analytic curves under the group of equilong transformations of the plane. Except for certain types, invariants always exist. The results are roughly analogous to Kasner's results for the conformal group.‡

In Chapter II, the use of Hessian line coordinates in the determination of a lineal element and in the equations of curves is first discussed. Then, by means of these coordinates, the equations of the n th evolute and the n th

* Cf. Scheffers, Isogonalkurven, Aquitangentalkurven, und komplexe Zahlen, *Mathematische Annalen*, bd. 60.

† Hessian line coordinates are completely described in Chap. II, 1.

‡ Cf. Kasner, Conformal classification of analytic arcs or elements; Poincaré's local problem of conformal geometry, *Trans. Amer. Math. Soc.*, vol. 16 (1915), pp. 333-349.

involute are derived. Expressions for the successive radii of curvature of an analytic curve in terms of the coefficients of the infinite series representing the curve are also obtained. Finally it is shown that if an equation be interpreted successively in Hessian line and polar point coordinates the equation represents a curve and its pedal respectively.

In Chapter III, the equilong transformations of the plane which are represented by the linear functions of the dual variable $u + jv$ are studied. The treatment is largely after the manner of treatment of the conformal transformation represented by the linear functions of the complex variable $x + iy$.^{*} It is seen that all linear transformations consist of certain particular transformations and their combinations. For the several types of motion, the effect on certain systems of curves, the systems of curves along which the motions may be assumed to take place, the invariants, the fixed lines, and the conditions are obtained. The simplest invariant of the regular analytic curve under the general linear transformation is obtained also.

The writer acknowledges with great pleasure his indebtedness to Professor Kasner for helpful suggestions and criticisms.

^{*} Cf. Cole, The Linear Functions of a Complex Variable, *Annals of Mathematics*, vol. 5.

CHAPTER I.

THE INVARIANTS OF IRREGULAR ANALYTIC CURVES UNDER THE GROUP OF EQUILONG TRANSFORMATIONS OF THE PLANE.

If the irregular analytic curve*

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \dots$$

is to go into the irregular analytic curve

$$V = A_q U^{q/p} + A_{q+1} U^{(q+1)/p} + A_{q+2} U^{(q+2)/p} + \dots,$$

p and q being positive integers, $p \geq 2$ and $q \geq p$, under the general equilong transformation

$$U = \sum_{r=1}^{\infty} a_r u^r,$$

$$V = \sum_{r=1}^{\infty} v r a_r u^{r-1} + \sum_{s=1}^{\infty} b_s u^s,$$

then the result obtained by eliminating U , V , and v among the four equations must be an identity in u or

$$\begin{aligned} & \sum_{k=0}^{\infty} \sum_{r=1}^{\infty} r a_r \alpha_{q+k} u^{[q+(r-1)p+k]/p} + \sum_{s=1}^{\infty} b_s u^s - \sum_{n=0}^{\infty} A_{q+n} \left[a_1^{(q+n)/p} u^{(q+n)/p} \right. \\ & + \sum_{r=1}^{\infty} \frac{\left(\frac{q+n}{p} \right) \left(\frac{q+n-p}{p} \right) \left(\frac{q+n-2p}{p} \right) \dots \left(\frac{q+n-(1-r)p}{p} \right)}{x! y! z! \dots} \\ & \left. \times a_1^{(q+n-rp)/p} a_2^x a_3^y a_4^z \dots u^{[q+n+(x+2y+3z+\dots)p]/p} \right] \equiv 0; \end{aligned}$$

$x, y, z, \dots$ being positive integers or zero satisfying the relation

$$x + y + z + \dots = r.$$

Suppose $p = 2$. If the coefficients of any number of the powers of u be each equated to zero, the number of the constants of the transformation present in the equations will always be found to equal or to exceed the number of equations. Hence there are no invariants.

* An irregular analytic curve (arc) is a curve which, in the neighborhood of a given line taken as the line $(0, 0)$ of the Hessian coordinate system, cannot be represented by setting v equal to a series in integral powers in u but can be represented by a series with fractional exponents.

Suppose $p > 2$. First, if neither q/p nor $(q+1)/p$ be an integer then from the equations formed by equating to zero the coefficients of $u^{q/p}$ and $u^{(q+1)/p}$,

$$a_1 \alpha_q - a_1^{q/p} A_q = 0,$$

$$a_1 \alpha_{q+1} - a_1^{(q+1)/p} A_{q+1} = 0,$$

the constant a_1 of the transformation may be eliminated and an invariant

$$\frac{\alpha_q^{q+1-p}}{\alpha_{q+1}^{q-p}} = J_{q+1}$$

obtained.

Secondly, if q/p be an integer, $(q+1)/p$ and $(q+2)/p$ will not be integers. From the equations formed by equating to zero the coefficients of $u^{(q+1)/p}$ and $u^{(q+2)/p}$,

$$a_1 \alpha_{q+1} - a_1^{(q+1)/p} A_{q+1} = 0,$$

$$a_1 \alpha_{q+2} - a_1^{(q+2)/p} A_{q+2} = 0,$$

the constant a_1 may be eliminated and an invariant

$$\frac{\alpha_{q+1}^{q+2-p}}{\alpha_{q+2}^{q+1-p}} = J_{q+2}$$

obtained.

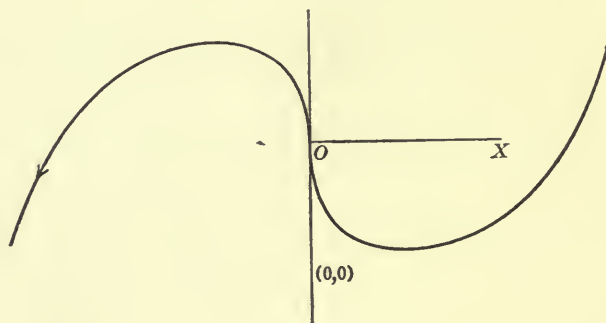


FIG. 1.

Thirdly, if $(q+1)/p$ be an integer, q/p and $(q+2)/p$ will not be integers. Then equating to zero the coefficients of $u^{q/p}$ and $u^{(q+2)/p}$ gives the two equations

$$a_1 \alpha_q - a_1^{q/p} A_q = 0,$$

$$a_1 \alpha_{q+2} - a_1^{(q+2)/p} A_{q+2} = 0,$$

from which a_1 may be eliminated and the invariant

$$\frac{\alpha_q^{q+2-p}}{\alpha_{q+2}^{q-p}} = J_{q+2}$$

obtained.

Hence every irregular analytic curve, except the curves

$$v = \alpha_q u^{q/2} + \alpha_{q+1} u^{(q+1)/2} + \alpha_{q+2} u^{(q+2)/2} + \dots,$$

has equilong invariants.*

Of the types which have no invariants, the form for $q = 3$,

$$v = u^{3/2},$$

is represented in Fig. 1.

* Regular curves are all reducible to the normal form $v = 0$ and hence have no equilong invariants. Cf. Kasner, *Conformal Geometry*, *Proc. Fifth Inter. Congress of Math.*, Cambridge, 1912, vol. 2, p. 85.

CHAPTER II.

SOME CURVES WITH EQUATIONS IN HESSIAN LINE COORDINATES.

1. HESSIAN LINE COORDINATES.

A plane curve is to be regarded as the envelope of a single infinity of oriented straight lines each of which is determined by the two Hessian coordinates (u, v) , v being the length of a perpendicular from a fixed point to the line (u, v) and u being the angle between the perpendicular (v) and a fixed line through the fixed point. On the perpendicular (v) , the distance from the fixed point to the line (u, v) is considered positive or negative according as the fixed point is on the left or on the right of the direction of the line (u, v) . The angle (u) is generated by rotating the fixed line until its positive sense coincides with the positive sense of the perpendicular (v) and is positive or negative according as the rotation is in the counter-clockwise or in the clockwise direction.

A lineal element of the plane is ordinarily determined by $(x, y, dy/dx)$, (x, y) being the cartesian coordinates of its point and dy/dx the tangent of the angle which its direction makes with the positive direction of the x -axis. It may also be determined by the quantities $(u, v, dv/du)$, (u, v) being the Hessian coordinates of its line (direction) and dv/du the distance along its line from the foot of the perpendicular (v) to its point. If $(x, y, dy/dx)$ and $(u, v, dv/du)$ represent the same lineal element, the following relations exist:*

$$\begin{aligned} u &= \tan^{-1} \frac{dy}{dx} - \frac{\pi}{2}, \\ v &= \frac{x \frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}, \\ \frac{dv}{du} &= \frac{x + y \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}. \end{aligned}$$

In general a plane curve is oriented and is assumed to have an equation of the form

* Cf. Scheffers, loc. cit., p. 520.

$$v = f(u).^*$$

The equation $u = k$ (a constant) represents a point at infinity which is, of course, regarded as a curve.

It is easily seen geometrically that the contact transformation

$$U = u + \beta_1,$$

$$V = v,$$

$$\frac{dV}{dU} = \frac{dv}{du},$$

is simply rotation of the plane about the fixed point $v = 0$ through an angle β_1 . It is clear also, geometrically, that the transformation

$$U = u,$$

$$V = v + \beta_2,$$

$$\frac{dV}{dU} = \frac{dv}{du},$$

is dilatation of the plane through the distance β_2 . In general, then, a curve, involute of some curve, E , remains an involute of that same curve, E , under the transformation

$$U = u + \beta_1,$$

$$V = v + \beta_2.$$

2. EVOLUTES AND INVOLUTES.

The n th evolute and the n th involute of a curve, $v = f(u)$, may be easily obtained.

Let the curve $v = f(u)$, the evolute of which is sought, be tangent to the line $AB(u, v)$ (Fig. 2) at B . BD , which is perpendicular to AB , will be tangent to the evolute at D . Let the coordinates of BD be (u_1, v_1) . Then

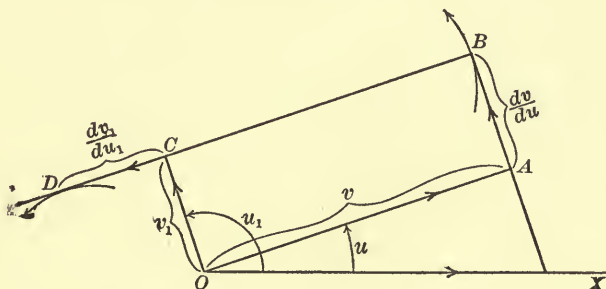


FIG. 2.

* This form is easily seen to be similar to the better known "magical equation of the tangent to a curve." Cf. Loria, Sp. Alg. u. Trans. Ebene Kurven (Gr. trans. by Schütte), bd. II, p. 255.

$$OC = AB$$

or

$$v_1 = \frac{dv}{du} = \frac{df(u)}{du} = f'(u).$$

Replacing u by $u_1 - \frac{\pi}{2}$,

$$v_1 = f' \left(u_1 - \frac{\pi}{2} \right)$$

is obtained. Hence the equation of the first evolute of the curve

$$v = f(u)$$

is

$$v = f' \left(u - \frac{\pi}{2} \right).$$

Since, however, the substitution of $u + \frac{\pi}{2}$ for u is equivalent simply to rotating a curve through the angle $-\pi/2$ about the point $v = 0$ of the plane, the equation of the first evolute of the curve

$$v = f(u)$$

may be taken to be

$$v = f'(u).$$

It follows that the n th evolute of a curve

$$v = f(u)$$

is

$$v = f^{(n)}(u),$$

where

$$f^{(n)}(u) = \frac{d^n f(u)}{du^n}.$$

To place the evolute in its natural position with respect to the curve, it should be rotated through the angle $n\pi/2$ or its equivalent.

The equation of the involute of a curve may be deduced independently of the results just obtained. It may also be simply derived by means of those results. If the equation of the involute of the curve

$$v = f(u)$$

is assumed to be

$$v = g(u),$$

then

$$g'(u) = f(u)$$

and

$$g(u) = \int f(u) du + \text{constant}.$$

Hence the equation of a first involute of the curve

$$v = f(u)$$

is

$$v = \int f(u)du + \text{constant.}$$

The variation of the constant through all possible values gives the ∞^1 involutes of the curve. In particular, the constant may take the value zero. Consequently it follows that the first involute of the curve

$$v = f(u)$$

may be written

$$v = \int f(u)du$$

and the n involute may be written

$$v = \int^{(n)} f(u) \overline{du}^n.$$

If the constant is not taken equal to zero for each successive involute, the equation of the n th involute is

$$v = \int^{(n)} f(u) \overline{du}^n + c_{n-1}u^{n-1} + c_{n-2}u^{n-2} + \dots + c_0,$$

the c 's being constants.

To place the n th involute in its natural position with respect to the curve it should be rotated through the angle $3n\pi/2$ or its equivalent.

The results just obtained may be used to find the involutes of a circle. If the equation of a circle be

$$v = a,$$

then the equation of the first involute of a circle is

$$v = au + b$$

and of the second involute

$$v = \frac{a}{2}u^2 + bu + c$$

and of the n th involute

$$v = \frac{a}{n!}u^n + \frac{b}{(n-1)!}u^{n-1} + \frac{c}{(n-2)!}u^{n-2} + \dots + k.$$

Hence an equation of the form

$$v = c_n u^n + c_{n-1} u^{n-1} + \dots + c_0$$

represents the n th involute of the circle

$$v = c_n \cdot n!.$$

3. RADII OF CURVATURE.

The radii of curvature of the successive evolutes of a curve (analytic arc) at a tangent of the curve are expressible conveniently in terms of the coefficients of the infinite series representing the arc, and conversely. Let the radius of curvature of the k th evolute be denoted by r_k , the radius of curvature of the curve being r_0 . Since

$$r_0 = \frac{ds}{du} = v + \frac{d^2v}{du^2},$$

then

$$r_1 = \frac{dr_0}{du} = \frac{d^2s}{du^2} = \frac{dv}{du} + \frac{d^3v}{du^3}$$

and

$$r_k = \frac{d^k v}{du^k} + \frac{d^{k+2}v}{du^{k+2}}.$$

An analytic arc is represented by the series

$$v = c_0 + c_1u + c_2u^2 + \dots + c_mu^m + \dots,$$

the c 's being constants. By rotation and translation* the arc may be so placed that the point of tangency is the fixed point and the normal through the point of tangency is the fixed line of the system of coordinates. Accordingly, an analytic arc may be represented by the infinite series

$$v = c_2u^2 + c_3u^3 + c_4u^4 + \dots + c_mu^m + \dots.$$

At the line $(0, 0)$,

$$\frac{dv}{du} = 0,$$

$$\frac{d^2v}{du^2} = 2c_2,$$

and

$$\frac{d^m v}{du^m} = m!c_m.$$

Hence

$$r_0 = \frac{d^2v}{du^2} = 2!c_2,$$

$$r_1 = \frac{d^3v}{du^3} = 3!c_3,$$

$$r_k = \frac{d^k v}{du^k} + \frac{d^{k+2}v}{du^{k+2}} = k!c_k + (k+2)!c_{k+2}.$$

* Translation ($X = x + \beta_1$; $Y = y + \beta_2$; $dY/dX = dy/dx$) is expressed by the equations $U = u$; $V = v + \beta_1 \cos u + \beta_2 \sin u$; $dV/dU = dv/du - \beta_1 \sin u + \beta_2 \cos u$.

If these equations be solved for the c 's, then

$$c_2 = \frac{\frac{d^2v}{du^2}}{2!} = \frac{r_0}{2!},$$

$$c_3 = \frac{\frac{d^3v}{du^3}}{3!} = \frac{r_1}{3!},$$

$$c_4 = \frac{\frac{d^4v}{du^4}}{4!} = \frac{r_2 - r_0}{4!},$$

$$c_5 = \frac{\frac{d^5v}{du^5}}{5!} = \frac{r_3 - r_1}{5!},$$

and

$$c_{n+2} = \frac{\frac{d^{n+2}v}{du^{n+2}}}{(n+2)!} = \frac{\sum_{p=1}^{p \leq n/2+1} (-1)^{p+1} r_{n+2-2p}}{(n+2)!},$$

in which p takes positive integral values.

By means of these results, the Cauchy test for the convergency of series may be interpreted geometrically. If the series

$$v = c_2 u^2 + \dots + c_k u^k + \dots$$

represent an analytic arc, then the superior limit, as $n \rightarrow \infty$, of

$$\sqrt[n+2]{\left| \frac{\sum_{p=1}^{p \leq (n/2)+1} (-1)^{p+1} r_{n+2-2p}}{(n+2)!} \right|}$$

is finite. If this condition be not fulfilled then the series is divergent and represents geometrically not an analytic curve but what Kasner* has called a "divergent differential element of infinite order."

4. PEDALS AND NEGATIVE PEDALS.

Suppose the equation of an oriented curve be expressed in polar coordinates (ρ, θ) . Let the coordinates be slightly modified by the following conditions. Let the direction on the radius vector (ρ) from the pole to a point of a curve be positive or negative according as the pole lies on the left or on the right of the positive direction of the curve at the point. Let the conditions which govern the sense of u also govern the sense of the vectorial angle θ . Let the positive direction on the polar subnormal $(d\rho/d\theta)$ be that which makes with

* Cf. *Bull. of the Am. Math. Soc.*, vol. XX, no. 10, p. 531.

the positive sense of the radius vector (ρ) an angle of $+\pi/2$. The three magnitudes (ρ , θ , $d\rho/d\theta$) may be used to determine a lineal element of the plane. The point of the element is fixed by (ρ , θ). The direction of the element is determined by $d\rho/d\theta$, being the direction perpendicular to the line (normal of the arc of the curve) joining the point (ρ , θ) to the extremity (other than the pole) of the polar subnormal and being determined in sense by the convention determining the sense of the radius vector (ρ).

Consider now the transformation

$$\begin{aligned}\theta &= u, \\ \rho &= v, \\ \frac{d\rho}{d\theta} &= \frac{dv}{du}.\end{aligned}$$

This is seen geometrically (Fig. 3) to be the foot point transformation, the element (u , v , dv/du) becoming the element (θ , ρ , $d\rho/d\theta$). The result is obtained analytically by expressing the transformation in the coordinates (x , y , dy/dx), representing the lineal element (u , v , dv/du), and (X , Y , dY/dX), representing the lineal element (θ , ρ , $d\rho/d\theta$). In this notation, the transformation is

$$\begin{aligned}\sqrt{X^2 + Y^2} &= \frac{x \frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}, & \tan^{-1} \frac{Y}{X} &= \tan^{-1} \frac{dy}{dx} - \frac{\pi}{2}, \\ \frac{\sqrt{X^2 + Y^2} \left(X + Y \frac{dY}{dX} \right)}{X \frac{dY}{dX} - Y} &= \frac{x + y \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}},\end{aligned}$$

or

$$\begin{aligned}X &= \pm \frac{\frac{dy}{dx} \left(x \frac{dy}{dx} - y \right)}{1 + \left(\frac{dy}{dx}\right)^2}, & Y &= \mp \frac{x \frac{dy}{dx} - y}{1 + \left(\frac{dy}{dx}\right)^2}, \\ \frac{dY}{dX} &= \frac{x \left(\frac{dy}{dx}\right)^2 - x - 2y \frac{dy}{dx}}{y \left(\frac{dy}{dx}\right)^2 - y + 2x \left(\frac{dy}{dx}\right)},\end{aligned}$$

which represents the foot point transformation or the foot point transformation followed by rotation through the angle π , according as the upper or lower signs are taken respectively.

Hence if a curve be expressed by the equation $v = f(u)$, in Hessian line coordinates, its pedal, with respect to the point $v = 0$, is expressed by the same equation $\rho = f(\theta)$, in polar point coordinates. Conversely, if a curve

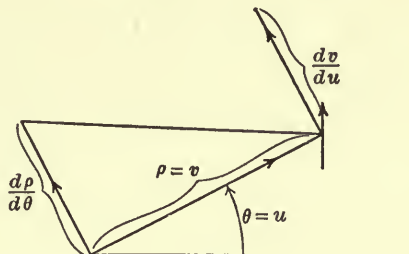


FIG. 3.

be expressed by the equation $\rho = f(\theta)$, in polar point coordinates, its first negative pedal with respect to the point $\rho = 0$, is expressed by the same equation, $v = f(u)$, in Hessian line coordinates.

For example, the equation

$$\rho = c_n \theta^n + c_{n-1} \theta^{n-1} + \dots + c_0$$

represents the pedal of the n th involute ($v = c_n u^n + c_{n-1} u^{n-1} + \dots + c_0$) of a circle ($v = n!c_n$) with respect to the center of the circle and the equation $v = a^n$ represents the first negative pedal of the equiangular spiral ($\rho = a^\theta$) with respect to its pole and represents, consequently, an equiangular spiral.

The beauty of Scheffers' treatment of equilong transformations is due to the combining of the Hessian line coordinates with the dual number $u + jv$ ($j^2 = 0$). If polar coordinates were combined with this dual number, an infinite group of point transformations would be obtained under which the difference of the polar subnormals (with respect to a fixed point) of two concurrent curves at their common point would be invariant.

CHAPTER III.

LINEAR EQUILONG TRANSFORMATIONS.

1. THE INTEGRAL LINEAR TRANSFORMATIONS.

(a) *The Operation* $W = w + \beta$.

The integral linear function, $W = U + jV$, of the dual variable,

$$w = u + jv,$$

is

$$W = \alpha w + \beta,$$

where α and β are dual numbers of the same form as W and w .

If $\alpha = 1$, the function becomes

$$W = w + \beta,$$

which represents the transformation

$$U + jV = u + jv + \beta_1 + j\beta_2$$

or

$$U = u + \beta_1,$$

$$V = v + \beta_2,$$

$$\frac{dV}{dU} = \frac{dv}{du}.*$$

First, let $\beta_1 \neq 0$ and $\beta_2 = 0$. Replacing the coordinates $(u, v, dv/du = v')$, which may be referred to as the Hessian coordinates of a lineal element, by the coordinates $(x, y, dy/dx = y')$, cartesian coordinates, the transformation takes the form

$$\tan^{-1} Y' - \frac{\pi}{2} = \tan^{-1} y' - \frac{\pi}{2} + \beta_1,$$

$$\frac{XY' - Y}{\sqrt{1 + Y'^2}} = \frac{xy' - y}{\sqrt{1 + y'^2}},$$

$$\frac{X + YY'}{\sqrt{1 + Y'^2}} = \frac{x + yy'}{\sqrt{1 + y'^2}},$$

or, solving for X , Y , and Y' ,

* The equilong transformations may be regarded as transformations of lines or of lineal elements in the plane.

$$X = x \cos \beta_1 - y \sin \beta_1,$$

$$Y = x \sin \beta_1 + y \cos \beta_1,$$

$$Y' = \frac{y' + \tan \beta_1}{1 - y' \cdot \tan \beta_1}.$$

These equations define rotation through the angle β_1 about the point $v = 0$. Every curve $v = k$, a circle with center at this point, is unchanged as a whole although its elements move in order around it.* The circles $v = k$ are therefore the curves of motion.†

Suppose, secondly, that $\beta_1 = 0$ and $\beta_2 \neq 0$. The transformation becomes

$$U = u,$$

$$V = v + \beta_2,$$

$$\frac{dV}{dU} = \frac{dv}{du}.$$

In cartesian coordinates, it is

$$\tan^{-1} Y' - \frac{\pi}{2} = \tan^{-1} y' - \frac{\pi}{2},$$

$$\frac{XY' - Y}{\sqrt{1 + Y'^2}} = \frac{xy' - y}{\sqrt{1 + y'^2}} + \beta_2,$$

$$\frac{X + YY'}{\sqrt{1 + Y'^2}} = \frac{x + yy'}{\sqrt{1 + y'^2}},$$

or

$$X = x + \frac{\beta_2 y'}{\sqrt{1 + y'^2}},$$

$$Y = y - \frac{\beta_2}{\sqrt{1 + y'^2}},$$

$$Y' = y',$$

which defines dilatation.

The orientation of the Hessian coordinates shows that the dilatation is in the direction which makes with the positive direction of the lineal element an

* An equilong transformation always turns every curve $u = k$, a point at infinity, into a curve $u = c$, another (or sometimes the same) point at infinity.

† The particular curve along which a particular line may be regarded as moving may be obtained by substituting the coordinates of the line in the equation of the family of curves, solving for the parameter, and then substituting the value obtained in the equation of the family of curves. If a line move from the position (a, b) to the position (A, B) along the curve $v = f(u)$, the distance passed over is $\int_a^A [f(u) + f''(u)] du$.

angle of $\mp \pi/2$ according as β_2 is positive or negative. Each circle, $v = k$, with center at the point $v = 0$, becomes another circle, $v = k + \beta_2$, with center at the same point. The curves of motion are the points at infinity, $u = k$.

The transformation

$$\begin{aligned} U &= u + \beta_1, \\ V &= v + \beta_2, \\ \frac{dV}{dU} &= \frac{dv}{du}, \end{aligned}$$

in which $\beta_1 \neq 0$ and $\beta_2 \neq 0$, is rotation and dilatation in succession. Since dv/du is unchanged by the transformation, the ∞^1 systems of curves

$$\frac{dv}{du} = c$$

are each unchanged. If the individual curves of one of the systems

$$v = cu + k$$

remain unchanged, then, after the substitution

$$\begin{aligned} u &= U - \beta_1, \\ v &= V - \beta_2, \end{aligned}$$

every curve

$$V = cU - c\beta_1 + k + \beta_2$$

must be identical with every curve

$$v = cu + k.$$

If this be so,

$$-c\beta_1 + k + \beta_2 = k$$

or

$$c = \frac{\beta_2}{\beta_1}.$$

Hence the curves

$$v = \frac{\beta_2}{\beta_1}u + k,$$

the ∞^1 first involutes of the circle $v = \beta_2/\beta_1$, are each unchanged as a whole and may be regarded as the curves along which the lines move under the transformation.

Since dv/du is invariant and since

$$\frac{d}{du} = \frac{d}{dU} \cdot \frac{dU}{du} = \frac{d}{dU} \cdot 1,$$

or

$$\frac{d}{du} = \frac{d}{dU},$$

every derivative $d^n v / du^n$ is invariant. Consequently the radius of curvature of every evolute of a curve is invariant. An involute of a curve is transformed therefore into another involute of the same curve. For rotation, one invariant is

$$v + \frac{d^2 v}{du^2} = r_0$$

and, for dilatation,

$$\frac{dv}{du} + \frac{d^3 v}{du^3} = r_1$$

which last is also an invariant of the rotation-dilatation transformation ($W = w + \beta$).*

(b) *The Operation* $W = \alpha w$.

If, in the integral linear function, $\beta = 0$, the transformation

$$W = U + jV = \alpha w = (\alpha_1 + j\alpha_2)(u + jv)$$

or

$$U = \alpha_1 u,$$

$$V = \alpha_1 v + \alpha_2 u,$$

$$\frac{dV}{dU} = \frac{dv}{du} + \frac{\alpha_2}{\alpha_1}$$

is obtained.

First, let $\alpha_2 = 0$. The transformation is then

$$U = \alpha_1 u,$$

$$V = \alpha_1 v,$$

$$\frac{dV}{dU} = \frac{dv}{du}.$$

The line $(0, 0)$ is fixed. A circle $v = k$, with center at the point $v = 0$, goes into a circle $v = \alpha_1 k$, with center at the same point. Since

$$\frac{v}{u} = \frac{\alpha_1 v}{\alpha_1 u} = \frac{V}{U},$$

every curve

$$\frac{v}{u} = k$$

is unchanged as a whole. Hence the curves of motion are $v = ku$, those ∞^1 first involutes of the ∞^1 circles $v = k$, which are all tangent to the fixed line $(0, 0)$.

* It is evident that Hessian line coordinates are convenient for the treatment of rotation and dilatation.

Secondly, let $\alpha_1 = 1$. The transformation is then

$$\begin{aligned} U &= u, \\ V &= v + \alpha_2 u, \\ \frac{dV}{dU} &= \frac{dv}{du} + \alpha_2. \end{aligned}$$

The points at infinity, $u = k$, are the curves of motion.

If, thirdly, $\alpha_1 \neq 1$ and $\alpha_2 \neq 0$, every circle $v = k$, with center at the point $v = 0$, is transformed into a first involute, $v = (\alpha_2/\alpha_1)u + \alpha_1 k$, of the circle $v = \alpha_2/\alpha_1$.

Let the equation of the curves along which the lines may be regarded as moving be

$$v = f(u).$$

By the transformation this equation must assume the form

$$V = f(U).$$

Hence the equations

$$v = f(u)$$

and

$$\alpha_1 v + \alpha_2 u = f(\alpha_1 u)$$

must be identical or the function f must satisfy the equation

$$\alpha_1 f(u) + \alpha_2 u = f(\alpha_1 u).$$

Differentiation gives

$$f'(u) - f'(\alpha_1 u) + \frac{\alpha_2}{\alpha_1} = 0.$$

Differentiation a second time gives

$$f''(u) - \alpha_1 f''(\alpha_1 u) = 0,$$

or

$$\alpha_1 = \frac{f''(u)}{f''(\alpha_1 u)},$$

and differentiation a third time,

$$f'''(u) - \alpha_1^2 f'''(\alpha_1 u) = 0.$$

By eliminating the coefficient α_1 between the last two equations, the equation

$$\frac{f'''(u)}{[f''(u)]^2} = \frac{f'''(\alpha_1 u)}{[f''(\alpha_1 u)]^2}$$

is obtained.

Hence

$$\frac{f'''(u)}{[f''(u)]^2} = \text{a constant.}$$

The solution of this differential equation, obtained by direct integration three times in succession, is

$$f(u) = a(u + b) \log(u + b) + cu + d.$$

The equation of the curves is then

$$v = a(u + b) \log(u + b) + cu + d,$$

in which the values of a , b , c , and d are to be determined. If this equation is to be unchanged in form by the transformation, then

$$a = \frac{\alpha_2}{\alpha_1 \log \alpha_1},$$

$$b = 0,$$

$$c \text{ is arbitrary,}$$

$$d = 0.$$

Hence the ∞' curves along which the lines may be regarded as moving under the transformation are represented by the equation

$$v = \frac{\alpha_2}{\alpha_1 \log \alpha_1} u \cdot \log u + cu.$$

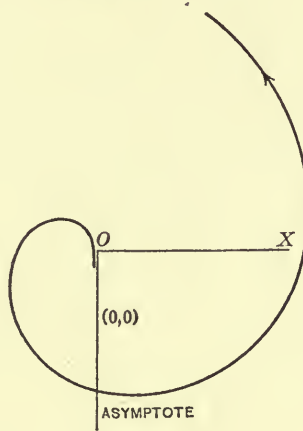


FIG. 4.

Consider the curve (Fig. 4)

$$v = \frac{\alpha_2}{\alpha_1 \log \alpha_1} u \cdot \log u + cu$$

or

$$u = ke^{\frac{\alpha_1 \log \alpha_1 v}{\alpha_2 u}}$$

It is of spiral shape since v increases as u increases. If

$$u = 0$$

then

$$v = 0$$

since

$$[u \cdot \log u]_{u=0} = 0$$

and

$$\left[\frac{dv}{du} \right]_{u=0} = \infty.$$

The line $(0, 0)$ is an asymptote.

The first evolute of this logarithmic* curve is

$$v = \frac{\alpha_2}{\alpha_1 \log \alpha_1} \log \left(u - \frac{\pi}{2} \right) + \left(c + \frac{\alpha_2}{\alpha_1 \log \alpha_1} \right),$$

which is the first negative pedal with respect to the point $v = 0$ of the logarithmic spiral of Varignon, the equation of which is, in polar coordinates,

$$\rho = - \left(\frac{-\alpha_2}{\alpha_1 \log \alpha_1} \right) \log \left(\theta - \frac{\pi}{2} \right) - \left(-c - \frac{\alpha_2}{\alpha_1 \log \alpha_1} \right).$$

The second evolute is

$$v(u - \pi) = \frac{\alpha_2}{\alpha_1 \log \alpha_1},$$

which is the first negative pedal of the hyperbolic spiral

$$\rho(\theta - \pi) = \frac{\alpha_2}{\alpha_1 \log \alpha_1},$$

with respect to its pole.

From the equations of the transformation, it follows that

$$\frac{d}{du} = \alpha_1 \frac{d}{dU}$$

and hence that

$$\frac{d^n v}{du^n} = \alpha_1^{n-1} \frac{d^n V}{dU^n} \quad [n = 2, 3, 4, \dots].$$

Therefore every expression of the form

$$\frac{\left[\frac{d^n v}{du^n} \right]^{m-1}}{\left[\frac{d^m v}{du^m} \right]^{n-1}} \quad [m = 2, 3, 4, \dots \neq n]$$

is invariant, the simplest being

$$\frac{\left[\frac{d^2 v}{du^2} \right]^2}{\frac{d^3 v}{du^3}} = J_3.$$

* The curve $v = a(u + b) \log(u + b) + cu + d$,— a , b , c , and d being constants,—will be referred to as a "logarithmic curve."

The invariants containing derivatives of even order only may be expressed in terms of the radii of curvature of the curve and its evolutes of even order. Consider an analytic curve in the neighborhood of the line $(0, 0)$. Then

$$\frac{d^2v}{du^2} = r_0,$$

$$\frac{d^4v}{du^4} = r_2 - r_0,$$

and

$$\frac{d^{2p}v}{du^{2p}} = \sum_{q=1}^{q=p} (-1)^{q+1} r_{2p-2q}. \quad [p = 1, 2, 3, \dots].$$

Consequently the invariants are

$$\frac{\left[\frac{d^{2p}v}{du^{2p}} \right]^{2p+1}}{\left[\frac{d^{2p+2}v}{du^{2p+2}} \right]^{2p-1}} = \frac{\left[\sum_{q=1}^{q=p} (-1)^{q+1} r_{2p-2q} \right]^{2p+1}}{\left[\sum_{q=1}^{q=p+1} (-1)^{q+1} r_{2p+2-2q} \right]^{2p-1}}.$$

The invariant of fourth order

$$\frac{\left[\frac{d^2v}{du^2} \right]^3}{\frac{d^4v}{du^4}}$$

may be written

$$\frac{r_0^3}{r_2 - r_0}.$$

(c) *The Operation* $W = \alpha w + \beta$.

The general integral linear function

$$W = \alpha w + \beta$$

represents the transformation

$$U = \alpha_1 u + \beta_1,$$

$$V = \alpha_1 v + \alpha_2 u + \beta_2,$$

$$\frac{dV}{dU} = \frac{dv}{du} + \frac{\alpha_2}{\alpha_1}.$$

Every circle $v = k$, with center at the point $v = 0$, goes into the curve

$$v = \frac{\alpha_2}{\alpha_1} u + \left(\beta_2 - \frac{\alpha_2 \beta_1}{\alpha_1} + \alpha_1 k \right),$$

a first involute of the circle $v = \alpha_2/\alpha_1$.

By the method employed in the case of the transformation $W = \alpha w$, the system of curves along which the lines may be regarded as moving is found to be

$$v = \frac{\alpha_2}{\alpha_1 \log \alpha_1} \left[u - \frac{\beta_1}{1 - \alpha_1} \right] \log \left[u - \frac{\beta_1}{1 - \alpha_1} \right] + cu + \frac{\alpha_2 \beta_1 + c \alpha_1 \beta_1 - c \beta_1 - \alpha_1 \beta_2 + \beta_2}{[1 - \alpha_1]^2},$$

a system of logarithmic curves with the line

$$\left(\frac{\beta_1}{1 - \alpha_1}, \frac{(1 - \alpha_1)\beta_2 + \alpha_2 \beta_1}{(1 - \alpha_1)^2} \right)$$

as asymptote.*

This system of curves may be obtained from the system of curves

$$v_1 = \frac{\alpha_2}{\alpha_1 \log \alpha_1} u_1 \cdot \log u_1 + cu_1,$$

for the transformation

$$W_1 = \alpha w_1,$$

by the rotation-dilatation transformation

$$w = w_1 + \frac{\beta}{1 - \alpha}.$$

Hence the transformation

$$W = \alpha w + \beta$$

may be regarded geometrically as the transformation

$$W_1 = \alpha w_1,$$

where

$$w_1 = w - \frac{\beta}{1 - \alpha}$$

and

$$W_1 = W - \frac{\beta}{1 - \alpha},$$

or in which the fixed line of the transformation is

$$\frac{\beta}{1 - \alpha} \text{ or } \left(\frac{\beta_1}{1 - \alpha_1}, \frac{(1 - \alpha_1)\beta_2 + \alpha_2 \beta_1}{(1 - \alpha_1)^2} \right).$$

The invariants of the transformation expressed by the general integral linear function $W = \alpha w + \beta$ are those which are common to the transformations expressed by the particular integral linear functions $W = w + \beta$ and

* If $\alpha_1 = 1$, the system of curves is

$$v = \frac{\alpha_2}{2\beta_1} u^2 + \frac{2\beta_2 - \alpha_2 \beta_1}{2\beta_1} u + c$$

and, if $\beta_1 = 0$ also, the system is $u = k$.

$W = \alpha w$ and are, consequently, those of the latter. The invariant of lowest order is

$$\frac{\left[\frac{d^2 v}{du^2} \right]^2}{\frac{d^3 v}{du^3}} = J_3.$$

The system of ∞^3 logarithmic curves

$$\frac{\left[\frac{d^2 v}{du^2} \right]^2}{\frac{d^3 v}{du^3}} = -a$$

or

$$v = a(u + b) \log(u + b) + cu + d,$$

b, c , and d being arbitrary, is invariant.

The equilong property of the transformation is easily deduced from

$$\frac{dV}{dU} = \frac{dv}{du} + \frac{\alpha_2}{\alpha_1}.$$

If two curves, tangent to the line (u, v) , have the distances $[dv/du]_1$ and $[dv/du]_2$ respectively, their transformed curves, tangent to the line (U, V) , will have the distances

$$\left[\frac{dV}{dU} \right]_1 = \left[\frac{dv}{du} \right]_1 + \frac{\alpha_2}{\alpha_1}$$

and

$$\left[\frac{dV}{dU} \right]_2 = \left[\frac{dv}{du} \right]_2 + \frac{\alpha_2}{\alpha_1}$$

respectively, and consequently

$$\left[\frac{dv}{du} \right]_2 - \left[\frac{dv}{du} \right]_1 = \left[\frac{dV}{dU} \right]_2 - \left[\frac{dV}{dU} \right]_1$$

and the distance between the points of tangency is invariant.

2. THE GENERAL LINEAR TRANSFORMATIONS.

(a) *The Operation* $W = \frac{1}{w}$.

The reciprocal function

$$W = \frac{1}{w}, *$$

represents the transformation

* The lines $u + jv$ where $u = 0$ are not considered. Throughout the entire article, the concept of a line or of lines at infinity is not considered and division by j is, of course, eliminated.

$$U + jV = \frac{1}{u + jv} = \frac{u - jv}{(u + jv)(u - jv)} = \frac{u - jv}{u^2},$$

or

$$U = \frac{1}{u},$$

$$V = -\frac{v}{u^2},$$

$$\frac{dV}{dU} = \frac{dv}{du} - 2\frac{v}{u}.$$

A circle $v = k$, with center at the point $v = 0$, is transformed into the curve $v = -ku^2$ which is a second involute of the circle $v = -2k$ and which is tangent to the line $(0, 0)$.

Since, from the equations of the transformation,

$$\frac{dv}{du} - \frac{v}{u} = \frac{dV}{dU} - \frac{V}{U},$$

the system of ∞^1 curves

$$\frac{dv}{du} - \frac{v}{u} = k$$

or

$$v = ku \cdot \log u + cu$$

is invariant. Each curve

$$v = ku \cdot \log u + cu$$

goes into the curve

$$v = ku \cdot \log u - cu,$$

the curve

$$v = ku \cdot \log u$$

going into itself.

Hence the ∞^1 logarithmic curves

$$v = ku \cdot \log u$$

are the curves along which the lines may be regarded as moving in order.

The transformation

$$U + jV = \frac{1}{u + jv}$$

is equivalent to the reverse equilong transformation

$$U + jV = \frac{1}{u - jv}$$

or

$$U = \frac{1}{u},$$

$$V = \frac{v}{u^2},$$

$$\frac{dV}{dU} = -\frac{dv}{du} + 2\frac{v}{u},$$

followed by the reverse equilong transformation

$$U + jV = u - jv$$

or

$$U = u,$$

$$V = -v,$$

$$\frac{dV}{dU} = -\frac{dv}{du}.$$

This last transformation is equilong symmetry with respect to the point $v = 0$. Equilong symmetry with respect to the curve $v = f(u)$ is defined in general by the equations*

$$U = u,$$

$$V = -v + 2f(u),$$

$$\frac{dV}{dU} = -\frac{dv}{du} + 2\frac{df(u)}{du}.$$

In cartesian coordinates, the transformation in this special case is

$$X = -x,$$

$$Y = -y,$$

$$\frac{dY}{dX} = \frac{dy}{dx},$$

which is rotation about the point $v = 0$ through the angle π . Since a lineal element and its symmetrical element are pointed in the same direction, the transformation has the effect of rotating every lineal element through the angle π and then reversing its direction. Every curve is rotated through an angle π and has its orientation reversed.

By the transformation

$$U = \frac{1}{u},$$

$$V = \frac{v}{u^2},$$

$$\frac{dV}{dU} = -\frac{dv}{du} + 2\frac{v}{u},$$

* This definition was given by Kasner, in a course at Columbia University, as a kind of analogue to conformal or Schwarzian symmetry.

a circle $v = k$ goes into a curve $v = ku^2$, which is a second involute of a circle $v = 2k$ and which is tangent to the line $(0, 0)$. Every curve

$$au + bv = 0,$$

which is a first involute of a circle $v = -a/b$ and which is tangent to the line $(0, 0)$, is invariant. A first involute, not tangent to $(0, 0)$,

$$au + bv + c = 0$$

is transformed into a second involute (spiral of Sturm or Norwich)

$$cu^2 + bv + au = 0,$$

tangent to $(0, 0)$. A second involute, tangent to $(0, 0)$,

$$cu^2 + bv + au = 0$$

goes into a first involute

$$au + bv + c = 0,$$

not tangent to $(0, 0)$. A second involute, not tangent to $(0, 0)$,

$$cu^2 + au + bv + d = 0$$

is transformed into a second involute

$$du^2 + au + bv + c = 0,$$

not tangent to $(0, 0)$. Each of the ∞^2 second involutes

$$cu^2 + au + bv + c = 0$$

is invariant.

By the transformation, the two points at infinity $u = \pm 1$ are rigid. Every line $(\pm 1, k)$ is fixed. A line

$$|u| < 1,$$

$$v = k$$

becomes a line

$$|U| > 1,$$

$$V > k$$

and a line

$$|u| > 1,$$

$$v = k$$

becomes a line

$$|U| < 1,$$

$$V < k.$$

The lines, of course, do not move in order along the curves of any system.

The transformation is of period 2 and is its own inverse as is also the transformation $W = 1/w$.

In the reciprocal transformation, $W = 1/w$,

$$\frac{dv}{du} - \frac{v}{u} = J_1$$

is invariant. Since

$$u \frac{d}{du} = -U \frac{d}{dU},$$

the invariant of order n may be written

$$\left\{ \left[u \frac{d}{du} \right]^{n-1} \left[\frac{dv}{du} - \frac{v}{u} \right] \right\}^2 = Jn,$$

in which the symbol $[u(d/du)]^{n-1}$ signifies the repetition $(n-1)$ times of the operation $u(d/du)$.

For the reciprocal transformation, the equilong property is easily shown to exist. If $(u, v, [dv/du]_1)$ and $(u, v, [dv/du]_2)$ be the coordinates of the lineal elements of two curves tangent to the line (u, v) and if $(U, V, [dV/dU]_1)$ and $(U, V, [dV/dU]_2)$ be the corresponding elements for the transformed curves tangent to the line (U, V) , then

$$\left[\frac{dV}{dU} \right]_2 - \frac{V}{U} = \left[\frac{dv}{du} \right]_2 - \frac{v}{u}$$

and

$$\left[\frac{dV}{dU} \right]_1 - \frac{V}{U} = \left[\frac{dv}{du} \right]_1 - \frac{v}{u}$$

and hence

$$\left[\frac{dV}{dU} \right]_2 - \left[\frac{dV}{dU} \right]_1 = \left[\frac{dv}{du} \right]_2 - \left[\frac{dv}{du} \right]_1.$$

(b) *Combinations of the Operations* $W = w + \beta$ and $W = \frac{1}{w}$.

If the lines (w_0) of a plane move along the curves, first involutes of a circle $v_0 = \beta_2/\beta_1$,

$$v_0 = \frac{\beta_2}{\beta_1} u_0 + k,$$

the corresponding lines $(w = 1/w_0)$ of a second plane, reciprocal to the first, will move along the curves

$$v = -ku^2 - \frac{\beta_2}{\beta_1} u.$$

These are second involutes of the circles $v = -2k$ and are tangent to the line $(0, 0)$ at the same point $([dv/du]_{u=0} = -\beta_2/\beta_1)$.

The equation of the transformation corresponding to motion along this system of curves may be determined in the following way. Let w_0 and w be

the initial positions and W_0 and W the final positions of lines in the first plane and in the reciprocal plane respectively. Since the motion in the first plane corresponds to the transformation

$$W_0 = w_0 + \beta$$

and since

$$w = \frac{1}{w_0}$$

and

$$W = \frac{1}{W_0},$$

then the motion in the reciprocal plane will be that of the transformation

$$\frac{1}{W} = \frac{1}{w} + \beta,$$

or

$$W = \frac{w}{\beta w + 1}.$$

If (in $\beta = \beta_1 + j\beta_2$) $\beta_1 = 0$, the motion is along the points at infinity $u = 1/k$, which are reciprocal to the points at infinity $u_0 = k$.

Suppose the system of second involutes to have the line (γ_1, γ_2) instead of the line $(0, 0)$ in common. The equation will then be

$$v - \gamma_2 = -k(u - \gamma_1)^2 - \frac{\beta_2}{\beta_1}(u - \gamma_1).$$

Subject the plane to the transformation

$$w_1 = w - \gamma.$$

The equation of the system of second involutes of circles takes the form

$$v_1 = -ku_1^2 - \frac{\beta_2}{\beta_1}u_1,$$

the line $(0, 0)$ being common to all the curves. Since motion along this system corresponds to the transformation

$$W_1 = \frac{w_1}{\beta w_1 + 1},$$

in which w_1 and W_1 represent respectively the initial and final positions of the lines, and since

$$w_1 = w - \gamma$$

and

$$W_1 = W - \gamma,$$

the transformation

$$W - \gamma = \frac{w - \gamma}{\beta(w - \gamma) + 1}$$

or

$$W = \frac{(1 + \beta\gamma)w - \beta\gamma^2}{\beta w - (1 - \beta\gamma)}$$

is obtained. This linear function then represents motion along a system of curves

$$v - \gamma_2 = -k(u - \gamma_1)^2 - \frac{\beta_2}{\beta_1}(u - \gamma_1),$$

each curve being a second involute of a circle ($v = -2k$) and being tangent to the line (γ_1, γ_2) at the same point ($[dv/du]_{u=\gamma_1} = -\beta_2/\beta_1$). Motion of this kind will be called parabolic.

If both numerator and denominator of the second member of the equation

$$W = \frac{(1 + \beta\gamma)w - \beta\gamma^2}{\beta w + (1 - \beta\gamma)}$$

be multiplied by an arbitrary number δ and if

$$\delta + \beta\delta\gamma = \alpha'$$

and

$$\beta\delta = \gamma'$$

and

$$\delta - \beta\delta\gamma = \delta',$$

then, dropping accents, the transformation will be expressed by the function

$$W = \frac{\alpha w - \frac{(\alpha - \delta)^2}{4\gamma}}{\gamma w + \delta}.$$

(c) *Combinations of the Operations* $W = \alpha w + \beta$ and $W = \frac{1}{w}$.

In the transformation

$$W_1 = \alpha w_1,$$

if (in $\alpha = \alpha_1 + j\alpha_2$) $\alpha_2 = 0$, the curves of motion are $v_1 = ku_1$, which are first involutes of the circles $v = k$ and which have the line $(0, 0)$ in common. If these curves be subjected to the transformation

$$w_0 = w_1 + \eta,$$

their equation will take the form

$$v_0 - \eta_2 = k(u_0 - \eta_1),$$

(η_1, η_2) becoming the common line. If lines (w_0) move along this system of

curves, the reciprocal lines ($w = 1/w_0$) will move along the system of curves

$$v = (k\eta_1 - \eta_2)u^2 - ku.$$

Motion along these curves, which are second involutes of the circles

$$v = 2(k\eta_1 - \eta_2)$$

and which have the lines $(0, 0)$ and $(1/\eta_1, -\eta_2/\eta_1^2)$ in common, will be called hyperbolic.

If, in the transformation ($W_1 = \alpha w_1$), $\alpha_1 = 1$, the curves of motion are the points at infinity $u_1 = k$. The lines ($w = 1/w_1$) reciprocal to the lines moving along these points will move along the points at infinity $u = 1/k$.

For the general transformation

$$W_1 = \alpha w_1,$$

the curves along which the lines may be regarded as moving are the ∞^1 logarithmic curves, c being an arbitrary constant,

$$v_1 = \frac{\alpha_2}{\alpha_1 \log \alpha_1} u_1 \cdot \log u_1 + cu_1.$$

If these curves be subjected to the transformation

$$w_0 = w_1 + \eta,$$

they become the ∞^1 logarithmic curves

$$v_0 - \eta_2 = \frac{\alpha_2}{\alpha_1 \log \alpha_1} (u_0 - \eta_1) \log (u_0 - \eta_1) + c(u_0 - \eta_1),^*$$

the reciprocal curves of which are, since $w = 1/w_0$, the ∞^1 curves

$$v = \frac{\alpha_2}{\alpha_1 \log \alpha_1} [\eta_1 u^2 - u] [\log (1 - \eta_1 u) - \log u] - cu + [c\eta_1 - \eta_2]u^2.$$

The line $(0, 0)$ is a common asymptote of this family of curves since if

$$u = 0$$

then

$$v = 0$$

and

$$\left[\frac{dv}{du} \right]_{u=0} = \infty.$$

The line $(1/\eta_1, -\eta_2/\eta_1^2)$, the reciprocal of the asymptote (η_1, η_2) of the system of logarithmic curves, is also a common asymptote of this family of curves.

Subject the plane to the transformation

* Motion along this system of curves corresponds to the transformation

$$W_0 = \alpha w_0 + \eta(1 - \alpha).$$

$$u' = u + \gamma_1,$$

$$v' = v + \gamma_2.$$

The asymptote $(0, 0)$ becomes the asymptote (γ_1, γ_2) and, if

$$\frac{1}{\eta_1} + \gamma_1 = \delta_1$$

and

$$-\frac{\eta_2}{\eta_1^2} + \gamma_2 = \delta_2,$$

the asymptote $(1/\eta_1, -\eta_2/\eta_1^2)$ becomes the asymptote (δ_1, δ_2) . The equation of the system of curves, c being an arbitrary constant, is, dropping accents,

$$\begin{aligned} v = & \frac{\alpha_2}{\alpha_1 \log \alpha_1} \left[\frac{(u - \gamma_1)(u - \delta_1)}{\gamma_1 - \delta_1} \right] [\log (u - \delta_1) - \log (u - \gamma_1)] \\ & + \left[-\frac{c(\gamma_1 - \delta_1) + (\gamma_2 - \delta_2)}{(\gamma_1 - \delta_1)^2} + \frac{\alpha_2}{\alpha_1 \log \alpha_1} \frac{\log (\gamma_1 - \delta_1)}{(\gamma_1 - \delta_1)} \right] u^2 \\ & + \left[\frac{c(\gamma_1 - \delta_1)(\gamma_1 + \delta_1) + 2\gamma_1(\gamma_2 - \delta_2)}{(\gamma_1 - \delta_1)^2} - \frac{\alpha_2}{\alpha_1 \log \alpha_1} \frac{(\gamma_1 + \delta_1) \log (\gamma_1 - \delta_1)}{(\gamma_1 - \delta_1)} \right] u \\ & + \left[\frac{(\gamma_1 - \delta_1)^2(c\gamma_1 + \gamma_2) - c\gamma_1^2(\gamma_1 - \delta_1) - \gamma_1^2(\gamma_2 - \delta_2)}{(\gamma_1 - \delta_1)^2} \right. \\ & \quad \left. + \frac{\alpha_2}{\alpha_1 \log \alpha_1} \frac{\gamma_1 \delta_1 \log (\gamma_1 - \delta_1)}{(\gamma_1 - \delta_1)} \right] .^* \end{aligned}$$

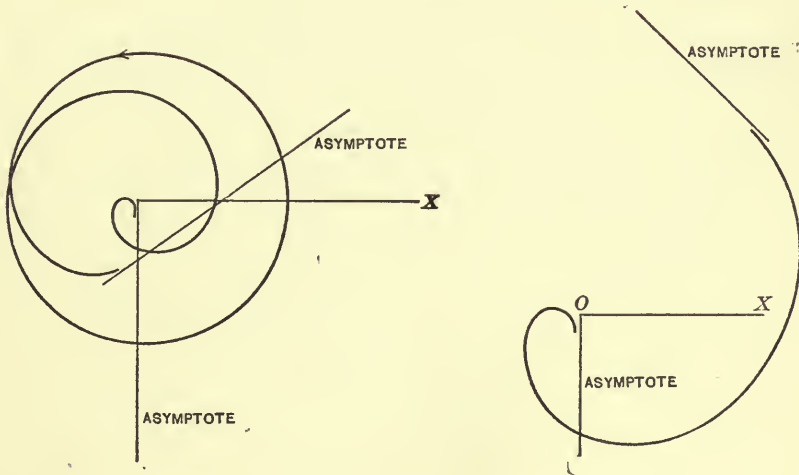


FIG. 5.

* A curve with this equation, $\alpha_2/\alpha_1 \log \alpha_1$, γ_1 , γ_2 , δ_1 , δ_2 , and c being constants, will be referred to as a curve (C) . In Fig. 5 two particular examples of the curve (C) are given. By a system of curves (C) will be meant the ∞^1 curves (C) obtained by taking definite values for the constants $\alpha_2/\alpha_1 \log \alpha_1$, γ_1 , γ_2 , δ_1 , and δ_2 and letting the constant c be arbitrary.

The equation of the transformation represented by motion along this system of curves may be derived in the following way. Suppose the lines are to move from the initial position w to the final position W . If the plane undergo the transformation

$$w_0 = w - \gamma,$$

$$W_0 = W - \gamma,$$

the curves, regarded as involutes, will be unchanged but the asymptote (γ_1, γ_2) will become the asymptote $(0, 0)$. As lines move along this system of curves, with one asymptote $(0, 0)$, from the positions (w_0) to the positions (W_0) , their reciprocal lines $(w_1 = 1/w_0)$ and $(W_1 = 1/W_0)$ will move according to the relation

$$W_1 = \alpha w_1 + \beta.$$

Hence

$$\frac{1}{W_0} = \alpha \frac{1}{w_0} + \beta$$

or

$$W_0 = \frac{w_0}{\beta w_0 + \alpha}$$

and

$$W - \gamma = \frac{w - \gamma}{\beta(w - \gamma) + \alpha}$$

or

$$W = \frac{(1 + \beta\gamma)w + (\alpha\gamma - \beta\gamma^2 - \gamma)}{\beta w + (\alpha - \beta\gamma)}.$$

The asymptote δ may be introduced instead of the quantity β . The transformation $w_0 = w - \gamma$ changes the line δ into the line $\delta - \gamma$. The motion

$$W_1 = \alpha w_1 + \beta$$

takes place about the fixed line $\beta/(1 - \alpha)$ which is the reciprocal of the line $\delta - \gamma$. Hence

$$\frac{\beta}{1 - \alpha} = \frac{1}{\delta - \gamma}$$

or

$$\beta = \frac{1 - \alpha}{\delta - \gamma}$$

and the transformation takes the form

$$W = \frac{(\delta - \alpha\gamma)w + (\alpha\delta\gamma - \delta\gamma)}{(1 - \alpha)w + (\alpha\delta - \gamma)}.$$

The lines γ and δ are the asymptotes of the system of curves (C) and α defines

the extent of the motion which is reciprocal to the given motion with respect to the asymptote.

The transformation may be written

$$W = \frac{\delta(w - \gamma) - \alpha\gamma(w - \delta)}{(w - \gamma) - \alpha(w - \delta)}.$$

The reverse transformation is then

$$w = \frac{\gamma(W - \delta) - \alpha\delta(W - \gamma)}{(W - \delta) - \alpha(W - \gamma)}.$$

If the first of these moves w to W , the second moves W back to w . Each transformation may be obtained from the other by interchanging γ and δ . Hence if the asymptotes of the motion be interchanged, a second motion equal and opposite to the first will be obtained.

The transformation is the general linear transformation and may be put in the form

$$W = \frac{\alpha'w + \beta'}{\gamma'w + \delta'}$$

or, dropping accents,

$$W = \frac{\alpha w + \beta}{\gamma w + \delta}.$$

(d) *The Fixed Lines and the Conditions for the Types of Motion.*

It has been shown that motions along the several systems of curves correspond to linear transformations of the variable $u + jv$. It will now be shown that all linear transformations give rise to the same systems of curves.

The transformation

$$W = \frac{\alpha w + \beta}{\gamma w + \delta}$$

in general leaves unchanged two lines of the plane which are the roots of the equation

$$\gamma w^2 + (\delta - \alpha)w - \beta = 0.$$

If the two roots are coincident and infinite, then

$$\gamma = 0$$

and

$$\delta - \alpha = 0$$

and the transformation reduces to

$$W = w + \frac{\beta}{\alpha}.$$

If the transformation is of this form, then it has no fixed lines in the finite plane.

If the two roots are coincident and finite, then

$$(\delta - \alpha)^2 + 4\beta\gamma = 0$$

and the transformation reduces to

$$W = \frac{\alpha w - \frac{(\delta - \alpha)^2}{4\gamma}}{\gamma w + \delta},$$

the motion corresponding to which has been called parabolic. A transformation of this form has two coincident lines fixed in the finite plane.

The fixed lines w_1 and w_2 , for parabolic motion, are

$$w_1 = w_2 = \frac{\frac{1}{2}(\alpha - \delta)}{\gamma}.$$

The condition

$$(\alpha - \delta)^2 + 4\beta\gamma = 0$$

may be written

$$\frac{[\frac{1}{2}(\alpha + \delta)]^2}{\alpha\delta - \beta\gamma} = 1.$$

If the motion takes place along points at infinity, then, from the equation for parabolic motion in the form

$$W = \frac{(1 + \beta\gamma)w + (-\beta\gamma^2)}{\beta w + (1 - \beta\gamma)},$$

$$\beta = 0 + j\beta_2.$$

If

$$\delta + \beta\delta\gamma = \alpha',$$

$$\beta\delta = \gamma',$$

and

$$\delta - \beta\delta\gamma = \delta',$$

then

$$\beta = \frac{2\gamma'}{\alpha' + \delta'} = 2 \frac{(\alpha_1' + \delta_1')\gamma_1' + j[(\alpha_1' + \delta_1')\gamma_2' - (\alpha_2' + \delta_2')\gamma_1']}{(\alpha_1' + \delta_1')^2}.$$

The further condition that parabolic motion take place along points at infinity is then, dropping accents, that, in the equation

$$W = \frac{\alpha w - \frac{(\delta - \alpha)^2}{4\gamma}}{\gamma w + \delta}, \quad \gamma_1 = 0.$$

If there is only one fixed line in the finite plane, then

$$\gamma = 0.$$

The transformation is

$$W = \frac{\alpha}{\delta} w + \frac{\beta}{\delta}.$$

Conversely, this transformation has one fixed line $\beta/(\delta - \alpha)$, which is the asymptote of the system of logarithmic curves.

If the two fixed lines are distinct and in the finite plane, one may be made to become the line $(0, 0)$ by the rotation-dilatation transformation, the other becoming at the same time some other line in the plane. By reciprocation, the line not $(0, 0)$ is transformed to some part of the plane. Motion about one line in the finite plane is motion along a system of logarithmic curves. Suppose the operations be reversed. The reciprocal of the motion along the system of logarithmic curves is motion along a system of curves (C) which has the line $(0, 0)$ and the line into which the second fixed line of the original transformation was transformed as asymptotes. Reversing the rotation-dilatation transformation, motion along the system of curves (C) which has the two distinct fixed lines in the finite plane as asymptotes is obtained. In general, the linear transformation which leaves two distinct lines fixed in the finite plane is represented by motion along a system of curves (C) . The fixed lines, w_1 and w_2 , of this motion are

$$w_1 = \frac{\frac{1}{2}(\alpha - \delta) + \sqrt{[\frac{1}{2}(\alpha - \delta)]^2 + \beta\gamma}}{\gamma}$$

and

$$w_2 = \frac{\frac{1}{2}(\alpha - \delta) - \sqrt{[\frac{1}{2}(\alpha - \delta)]^2 + \beta\gamma}}{\gamma}.$$

To determine the conditions for the hyperbolic case, the equation of the transformation in terms of the fixed lines and the quantity α , now called ϵ ,

$$W = \frac{(w_2 - \epsilon w_1)w + w_1 w_2 (\epsilon - 1)}{(1 - \epsilon)w + \epsilon w_2 - w_1},$$

is used. If this transformation is identical with the transformation

$$W = \frac{\alpha w + \beta}{\gamma w + \delta},$$

then, k being a constant,

$$k(w_2 - \epsilon w_1) = \alpha,$$

$$k w_1 w_2 (\epsilon - 1) = \beta,$$

$$k(1 - \epsilon) = \gamma,$$

$$k(\epsilon w_2 - w_1) = \delta.$$

Then

$$\frac{w_2 - \epsilon w_1}{\epsilon w_2 - w_1} = \frac{\alpha}{\delta}$$

and

$$\begin{aligned}\epsilon &= \epsilon_1 + j\epsilon_2 = \frac{\alpha + \delta + \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}}{\alpha + \delta - \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}} \\ &= \frac{[(\alpha + \delta) + \sqrt{(\alpha + \delta)^2 - 4(\alpha\delta - \beta\gamma)}]^2}{4(\alpha\delta - \beta\gamma)}.\end{aligned}$$

If

$$\epsilon_2 = 0,$$

then

$$\frac{[\frac{1}{2}(\alpha_1 + \delta_1)]^2}{\alpha_1\delta_1 - \beta_1\gamma_1} > 1$$

and

$$(\alpha_2 + \delta_2)(\alpha_1\delta_1 - \beta_1\gamma_1) - \frac{1}{2}(\alpha_1 + \delta_1)(\alpha_1\delta_2 + \alpha_2\delta_1 - \beta_2\gamma_1 - \beta_1\gamma_2) = 0.$$

The condition for hyperbolic motion is then that, in the transformation

$$W = \frac{\alpha w + \beta}{\gamma w + \delta},$$

$$\frac{[\frac{1}{2}(\alpha + \delta)]^2}{\alpha\delta - \beta\gamma} > 1.$$

For the case in which the motion is along points at infinity, the form of the transformation

$$W = \frac{(1 + \beta\gamma)w + (\alpha\gamma - \beta\gamma^2 - \gamma)}{\beta w + (\alpha - \beta\gamma)}$$

is used. If this transformation is identical with the transformation

$$W = \frac{\alpha'w + \beta'}{\gamma'w + \delta'},$$

then

$$\delta(1 + \beta\gamma) = \alpha',$$

$$\delta(\alpha\gamma - \beta\gamma^2 - \gamma) = \beta',$$

$$\delta\beta = \gamma',$$

$$\delta(\alpha - \beta\gamma) = \delta',$$

and

$$\frac{\alpha - 1}{(\alpha + 1)^2} = \frac{\alpha'\delta' - \beta'\gamma'}{(\alpha' + \delta')^2}.$$

If α is of the form $1 + j\alpha_2$, then

$$j\alpha_2 = \frac{\alpha'\delta' - \beta'\gamma'}{[\frac{1}{2}(\alpha' + \delta')]^2}.$$

For the motion along points at infinity the condition is, dropping accents and thereby taking the transformation to be

$$W = \frac{\alpha w + \beta}{\gamma w + \delta},$$

that

$$\frac{\alpha\delta - \beta\gamma}{[\frac{1}{2}(\alpha + \delta)]^2}$$

contains j .

The general transformation

$$W = \frac{\alpha w + \beta}{\gamma w + \delta}$$

contains three essential constants of the form $\eta_1 + j\eta_2$. There are then ∞^6 such real transformations.

In the general transformation, four constants determine the asymptotes, one constant determines the system of curves (C), and one constant determines the extent of the motion on the system.

One condition determines hyperbolic motion. Four constants determine the system of curves and one constant determines the extent of the motion.

Two conditions are required for parabolic motion. Two constants determine the common element of the system of second involutes of circles, one constant determines the distance to the common point of tangency, and one constant determines the extent of the motion.

All linear transformations, being obtainable from combinations of the transformations $W = w + \beta$, $W = \alpha w$, and $W = 1/w$, of course possess the equilong property.

(e) *The Simplest Invariant of a Regular Analytic Curve under the General Linear Transformation.*

The invariant of lowest order for the general linear transformation will next be derived. All the linear transformations which leave the line $(0, 0)$ unchanged are represented by the equation

$$W = \frac{\alpha w}{\gamma w + 1} = \alpha w(1 + \gamma w)^{-1}.$$

Expanding the right-hand member, the transformations may be written

$$U = \sum_{m=0}^{\infty} \alpha_1(-\gamma_1)^m u^{m+1},$$

$$V = \sum_{n=0}^{\infty} [v(n+1)\alpha_1(-\gamma_1)^n - (n\alpha_1(-\gamma_1)^{n-1}\gamma_2 - \alpha_2(-\gamma_1)^n)u]u^n.$$

If such a transformation turn the analytic curve

$$v = \sum_{k=1}^{\infty} b_k u^k$$

into the analytic curve

$$V = \sum_{k=1}^{\infty} B_k U^k,$$

then the result obtained by eliminating U , V , and v among the four equations is an identity with respect to u or

$$\sum_{p=1}^{\infty} \left\{ \sum_{n=0}^{p-1} b_{p-n}(n+1)\alpha_1(-\gamma_1)^n - (p-1)\alpha_1(-\gamma_1)^{p-2}\gamma_2 + \alpha_2(-\gamma_1)^{p-1} \right. \\ \left. - \sum_{s=1}^p B_s \alpha_1^s \frac{s!}{x_0! \cdots x_t!} (-\gamma_1)^{0x_0} \cdots (-\gamma_1)^{tx_t} \right\} u^p \equiv 0,$$

$x_0, \dots, x_t$ being positive integers or zero satisfying the relations

$$x_1 + 2x_2 + \cdots tx_t = p - s,$$

$$x_0 + x_1 + \cdots x_t = s.$$

If the coefficients of the first five powers of u be each equated to zero, the constants $(\alpha_1, \alpha_2, \gamma_1, \gamma_2)$ of the transformation may be eliminated with the result

$$\frac{b_3^3}{b_4^2 - b_3b_5} = \frac{B_3^3}{B_4^2 - B_3B_5}.$$

The left hand member of this equation is therefore invariant under the transformations. In terms of the derivatives of the curve $v = \sum_{k=1}^{\infty} b_k u_k$ at the line $(0, 0)$, this invariant has the form

$$J_5 = \frac{\left(\frac{d^3v}{du^3}\right)^3}{5\left(\frac{d^4v}{du^4}\right)^2 - 4\left(\frac{d^3v}{du^3}\right)\left(\frac{d^5v}{du^5}\right)}.$$

The transformations $W = w + \beta$ and $W = 1/w$ are not included in the transformations $W = \alpha w(1 + \gamma w)^{-1}$. For the transformations $W = w + \beta$ and $W = 1/w$, direct computation shows that J_5 is invariant.

Hence the invariant of lowest order for the general linear transformation $W = (\alpha w + \beta)(\gamma w + \delta)^{-1}$ is

$$J_5 = \frac{\left(\frac{d^3v}{du^3}\right)^3}{5\left(\frac{d^4v}{du^4}\right)^2 - 4\left(\frac{d^3v}{du^3}\right)\left(\frac{d^5v}{du^5}\right)}.$$

A family of curves characterized by the property

$$J_5 = \frac{1}{k} \text{ (a constant)}$$

is invariant under the general linear transformation. By solving the differential equation

$$\frac{\left(\frac{d^3v}{du^3}\right)^3}{5\left(\frac{d^4v}{du^4}\right)^2 - 4\left(\frac{d^3v}{du^3}\right)\left(\frac{d^5v}{du^5}\right)} = \frac{1}{k},$$

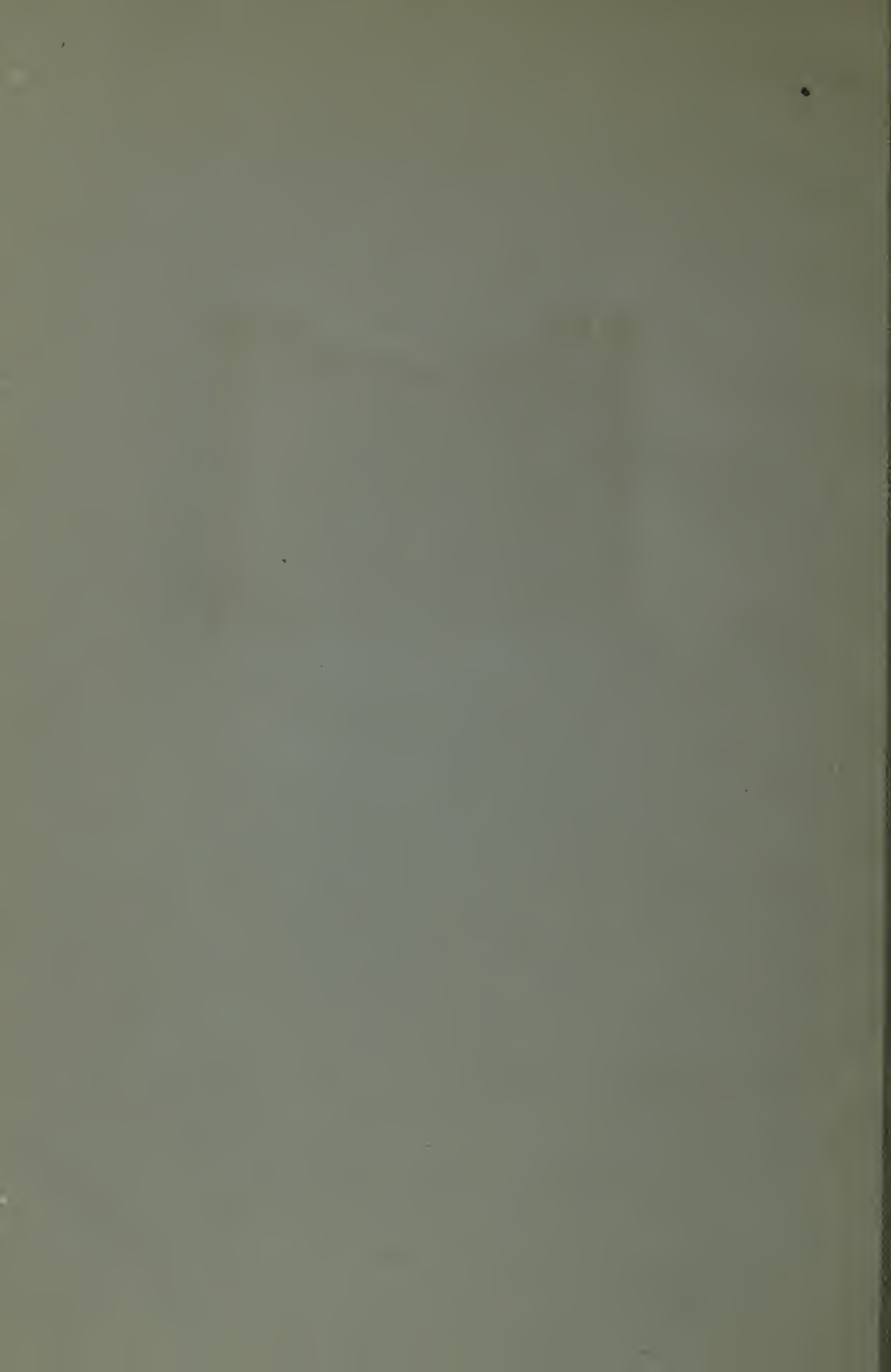
the ∞^5 curves of the family are found to be

$$v = \frac{a}{2k\sqrt{-k}} \left[(u+b)^2 + \frac{16k}{a^2} \right] \left[\log \left(u + b - \frac{4}{a}\sqrt{-k} \right) - \log \left(u + b + \frac{4}{a}\sqrt{-k} \right) \right] \\ + \frac{128c}{a^2} u^2 + \frac{256}{a^2} \left[d + \frac{a^2}{64k} \right] u + e,$$

a , b , c , d , and e being arbitrary constants.

VITA.

Paul Henry Linehan was born in Boston, Massachusetts, January 15, 1879. He attended the public schools of his native city and Harvard University, from which he received the degree of Bachelor of Arts in 1902. Since February, 1903, he has been, successively, tutor and instructor in mathematics in the College of the City of New York.



GABBI
L5

327802

Lindholm

UNIVERSITY OF CALIFORNIA LIBRARY

